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A SHALLOW SOLID BODY APPROACHING THE INTERFACE OF TWO MEDIA

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The plane unsteady problem of vertical motion of a shallow undeformable contour in a two-layer fluid is considered. At the initial moment, the body is in the upper fluid layer, and both the depth of the lower layer and the body–interface distance are assumed to be small compared with the characteristic linear size of the body. The body then begins to move vertically toward the interface. We consider the problem within the framework of two-layer shallow-water theory with emphasis on the possibility of and the conditions for the appearance of mixing zones. A model that describes such a flow is proposed and analyzed numerically.

The problem of a body approaching the interface of two media and its subsequent penetration into a lower fluid is of both theoretical and practical interest. This interest was aroused in the early 1960s and is connected with the pioneer investigation performed by Chuang [1] who demonstrated the role of air during the fall of a body on the surface of water and showed that before a falling body contacts the fluid, the free boundary has already been deformed by an air flow pushed by the body. It has been revealed that in the case of fall of a flat-bottom body, the deflection of the free boundary is so significant that a cavity is formed, which is closed at the moment of contact of the body and the fluid. The effect of the trapped air in the problem of fall of a body on the surface of water was later studied by many authors (see, e.g., [2, 3]). However, in all theoretical and numerical investigations, water and air were regarded as immiscible media with a distinct interface between them. This assumption is not always true from the physical viewpoint. The point is that as the distance between the bottom of the body and the interface decreases, the velocity of air outflow from beneath the bottom increases, which leads to the instability of the interface and the formation of air–water mixing zones [4]. Such zones can be formed only in regions in which the outflow velocity is sufficiently large [5]. At the moment of cavity closing, the body is in contact with an air–water mixture rather than with water. The air continues to flow out from the cavity through the mixing layer, giving rise to its growth and water spraying.

The goal of the present paper is to find conditions under which mixing zones appear and to estimate the dimensions and positions of these zones. The air motion in a thin layer between the liquid free surface and a shallow solid body approaching the liquid is usually considered within the framework of the one-dimensional approximation, and a liquid flow is usually considered within the framework of the linear approximation. Before analyzing the general nonlinear problem of body motion in a two-layer fluid, it is useful to examine in detail the simplest case of a thin lower layer using a shallow-water approximation for two-layer flows. The importance of the shallow-water theory for investigation of the general problem of unsteady fluid motion in the presence of a free boundary or the interfaces of layers of different density is well known [6, 7]. An analysis of the problem within the framework of the shallow-water theory will give us ideas for an adequate formulation of the problem in the general case.

Formulation of the Problem. We consider the vertical motion of a shallow symmetric contour in a two-layer fluid (Fig. 1). The body and the fluid are at rest until a certain moment of time which is regarded as the initial one. The point at which the symmetry line of the body intersects the layer interface is regarded as the origin of the Cartesian coordinate system $x'Oy'$. Here and below, the dimensional variables are primed. The lower layer is bounded from below by a horizontal undeformable bottom. At the initial moment ($t' = 0$), the body starts to move vertically downward with velocity V. We shall determine the velocity fields $\mathbf{u}^{+}(\mathbf{x}', t')$

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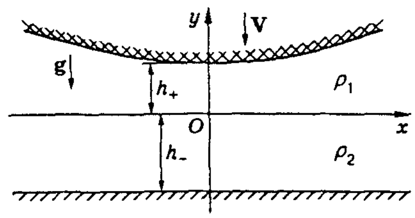

Fig. 1

and $\mathbf{u}^-(\mathbf{x}', t')$, the pressures $p'_+(\mathbf{x}', t')$ and $p'_-(\mathbf{x}', t')$ in the upper and lower layers, and the evolution of the interface $y' = \zeta'(x', t')$ under the following assumptions: the fluid is ideal and incompressible and the fluid flow is irrotational. Hereinafter, the plus and minus signs refer to the upper and lower layers, respectively. The body–fluid system is in the gravity field with acceleration of gravity g. Capillary forces with surface-tension coefficient σ act at the layer interface. For $t' > 0$, the body's position is determined by the equation

$$y' = h_+ f(x'/L) - Vt' + h_+,$$

where L is the characteristic linear size related to the geometry of the body, the dimensionless function $f(x)$ describes the body shape, h_+ is the initial distance from the body top to the interface, and h_- is the initial depth of the lower layer $[h_+/h_- = O(1)]$. The characteristic horizontal linear scale of the process L is assumed to be considerably greater than the corresponding vertical scale h_+, i.e., $\varepsilon = h_+/L \ll 1$. This points to the possibility of using the two-layer shallow-water theory for flow description. The function $f(x)$ is such that $f(0) = 0$, $f'(0) = 0$, and $x/f(x) \to 0$ at $x \to \infty$ and $f(-x) = f(x)$.

Below, we use dimensionless variables which are introduced as follows:

$$x' = Lx, \quad y' = h_+ y, \quad t' = h_+ t/V, \quad v' = Vv, \quad u' = V\varepsilon^{-1}u,$$
$$p'_+ = \rho_+ V^2 \varepsilon^{-2} p_+, \quad p'_- = \rho_- V^2 \varepsilon^{-2} p_-, \quad \zeta' = h_+ \zeta.$$

Here ρ_+ and ρ_- are the densities of the upper and lower media, respectively, and $u(x, y, t)$ and $v(x, y, t)$ are the horizontal and vertical components of the velocity vector $\mathbf{u} = (u, v)$.

It is convenient to describe the fluid flow in terms of the stream function $\psi(x, y, t)$ and a new desired function $h(x, t)$ such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad h(x, t) = -\int_0^x \zeta(\xi, t)\, d\xi. \tag{1}$$

The equations of motion and the boundary conditions take, in dimensionless variables, the following forms:

$$\varepsilon^2 \psi^+_{xx} + \psi^+_{yy} = 0 \quad (-h_x(x, t) < y < f(x) - t + 1),$$
$$\varepsilon^2 \psi^-_{xx} + \psi^-_{yy} = 0 \quad (-a < y < -h_x(x, t)),$$
$$p_- = \mu p_+ + \beta \frac{h_{xxx}}{(1 + \varepsilon^2 h_{xx}^2)^{3/2}}, \quad \psi^- = \psi^+ = h_t(x, t) \quad [y = -h_x(x, t)], \tag{2}$$
$$\psi^+ = x \quad (y = f(x) - t + 1), \quad \psi^- = 0 \quad (y = -a),$$
$$\psi, |\nabla \psi|, h_x \to 0 \quad (|x| \to \infty, t > 0), \quad \psi = 0, \quad h = 0 \quad (t < 0),$$

where $\mu = \rho_+/\rho_-$, $0 < \mu < 1$, $a = h_-/h_+$, and $\beta = \sigma h_+^3/(\rho_- V^2 L^4)$. The pressure p is related to the stream function by the following momentum equations, which are of the same form both for the upper and lower layers:

$$u_t + uu_x + vu_y + p_x = 0, \quad \varepsilon[v_t + uv_x + vv_y] + p_y + \alpha = 0, \tag{3}$$

where $\alpha = \varepsilon^3(gL/V^2)$. In each layer, the condition that the flow is potential yields

$$u_y = \varepsilon^2 v_x. \tag{4}$$

We shall find an approximate solution of problem (1)–(4) for $t > 0$ and $\varepsilon \ll 1$. Note that within the framework of the incompressible-fluid model, the fluid and interface velocities are set up instantaneously for the impulsive start of the body. These velocities are calculated using the Sedov impact theory [8] and determine the initial conditions for system (2) at $t = +0$.

Shallow-Water Approximation. The solution of problem (1)–(4) is searched for as a series in integral powers of the parameter ε. Assuming that $\beta = O(1)$ and $\alpha = O(1)$ for $\varepsilon \to 0$, we obtain

$$\begin{gathered} u(x,y,t) = U(x,t) + O(\varepsilon^2), \quad p(x,y,t) = -\alpha y + P(x,t) + O(\varepsilon), \\ \psi^+(x,y,t) = x + U^+(x,t)(y + t - f(x) - 1) + O(\varepsilon^2), \\ \psi^-(x,y,t) = U^-(x,t)(y+a) + O(\varepsilon^2), \quad h(x,t) = h^{(0)}(x,t) + O(\varepsilon^2). \end{gathered} \tag{5}$$

Here

$$\begin{gathered} U(x,t) = \frac{Q(x,t)}{H(x,t)}, \qquad Q_+ = x - h_t^{(0)}, \qquad Q_- = h_t^{(0)}, \\ H_+ = f(x) - t + 1 + h_x^{(0)}, \qquad H_- = a - h_x^{(0)}. \end{gathered} \tag{6}$$

With allowance for (3)–(5), the dynamic condition at the interface of the media ensures that

$$U_t^- + U^- U_x^- = \mu[U_t^+ + U^+ U_x^+] + \alpha(1-\mu) h_{xx}^{(0)} + \beta h_{xxxx}^{(0)} + O(\varepsilon^2 + \beta\varepsilon^2). \tag{7}$$

Replacing $U^+(x,t)$ and $U^-(x,t)$ by their relations (6) in (7), we obtain one quasi-linear differential equation relative to the function $h^{(0)}(x,t)$. In what follows, the subscript (0) is omitted. In a first approximation, we have

$$A h_{tt} + 2B h_{xt} + C h_{xx} + \beta h_{xxxx} = D, \tag{8}$$

where

$$\begin{gathered} A = H_+^2 H_-^2 (H_+ + \mu H_-), \quad B = H_+ H_- (H_+^2 Q_- + \mu H_-^2 Q_+), \\ C = H_+^3 Q_-^2 + \mu H_-^3 Q_+^2 - (1-\mu)\alpha H_+^3 H_-^3, \quad D = \mu H_-^3 Q_+ (2H_+ - f'(x) Q_+). \end{gathered} \tag{9}$$

By virtue of flow symmetry, Eq. (8) is considered only for $x > 0$. The boundary conditions for Eq. (8) are of the form

$$h(0,t) = 0, \quad h_{xx}(0,t) = 0, \quad h(x,t) \to 0, \quad h_x(x,t) \to 0 \quad (x \to \infty), \tag{10}$$

and the initial conditions are as follows:

$$h(x,+0) = 0, \quad h_t(x,+0) = b(x). \tag{11}$$

To define the function $b(x)$, we integrate Eq. (7) over time from $-\delta$ to δ and take the limit as $\delta \to 0$. We find $U^-(x,+0) = \mu U^+(x,+0)$. With allowance for (6), we obtain

$$b(x) = \frac{a\mu x}{f(x) + 1 + a\mu}. \tag{12}$$

Problem (8)–(12) contains four parameters a, μ, α, and β, the importance of each parameter being dependent on its magnitude.

In deriving Eq. (7), we can ignore the fluid viscosity under the condition that

$$V \ll \max(\nu^+, \nu^-)/h_+, \tag{13}$$

where ν is the kinematic viscosity coefficient. The contribution of the surface-tension forces will be of the order of the terms ignored in (7) under the condition $\beta = O(\varepsilon^2)$. If $\beta = O(1)$, these forces play the major role. Similar conditions for gravity are as follows: $\alpha(1-\mu)=O(\varepsilon^2)$ and $\alpha(1-\mu)=O(1)$. This means that Eq. (8) corresponds formally to the initial equation with accuracy up to $O(\varepsilon^2)$ if the conditions

$$V \ll \sqrt{\frac{\sigma h_+}{\rho_- L^2}}, \tag{14}$$

$$V \ll \sqrt{g h_+} \tag{15}$$

are satisfied. If condition (14) is not satisfied, one should set $\beta=0$ in (8), whereas if condition (15) is not satisfied, one should set $\alpha=0$. On the other hand, the surface-tension and gravity forces play a key role in the problem of body motion to the interface if $\beta=O(1)$ and $\alpha(1-\mu)=O(1)$, respectively.

We explain the meaning of these conditions by way of example. Let a parabolic contour with the radius of curvature $R=100$ m at its top be initially in air at a distance $h_+=1$ cm from the free boundary of a water layer. In this case,we have $L=\sqrt{Rh_+}=1$ m and $\varepsilon=0.01$. Unlike other effects, water viscosity can be ignored if the body velocity is substantially greater than 1 mm/sec, which follows from (13). Inequalities (14) and (15) give $V \ll 0.85$ mm/sec and $V \ll 31$ cm/sec, respectively. This implies that in the cases of practical importance where the impact velocity is approximately a few centimeters per second or higher, the viscosity and surface-tension forces can be ignored. If the impact velocity is of the order of a meter per second or higher, the gravity effects can be disregarded.

Qualitative Analysis of the Model. We consider a range of body velocities such that $\beta \ll 1$ and $\alpha(1-\mu)=O(1)$. In this case, Eq. (8) is the inhomogeneous quasi-linear second-order differential equation. The type of this equation is determined by the sign of the relation $\Delta(x,t,\alpha,\mu,a)=B^2-AC$. For $\Delta<0$, the equation is of elliptic type and, for $\Delta>0$, it is of hyperbolic type. The quantity Δ depends not only on the parameters α, μ, and a, but also on the independent variables x and t, and, therefore, the type of the equation can change with time. Calculations yield

$$\Delta(x,t,\alpha,\mu,a)=(H_+H_-)^5(\alpha(1-\mu)(H_+ + \mu H_-)-\mu(U^+ - U^-)^2). \tag{16}$$

The ellipticity regions with $\Delta<0$ appear where the velocity shear $|U^+ - U^-|$ is large. The velocity shear is small near the symmetry axis $[U^+(0,t)=U^-(0,t)=0]$, and $\Delta>0$ for any time. The fluid velocity decays with distance from the symmetry axis ($|x|\to\infty$), and, hence, $\Delta>0$. Thus, the ellipticity regions of Eq. (8) are separated from the symmetry axis for $\beta=0$ and are of finite dimensions. Let us determine the restrictions on the problem parameters at which the equation is of parabolic type at the initial moment ($t=+0$). The condition $\Delta(x,+0,\alpha,\mu,a)>0$ yields

$$\alpha > \mu(1-\mu)\max_{0<x<\infty}\left(\frac{x^2}{(f(x)+1+\mu a)^3}\right). \tag{17}$$

For a parabolic contour $[f(x)=x^2/2]$, we have

$$\alpha > \frac{8}{27}\frac{\mu(1-\mu)}{(\mu a+1)^2}. \tag{18}$$

In the case where $a=1$, $h_+=1$ cm, $R=100$ m, and $\mu=0.001$ (water–air system), the latter inequality leads to the restriction on the body velocity $V \leqslant 18$ cm/sec with satisfaction of which the initial data (11) and (12) lie within the hyperbolicity region of Eq. (8) for $\beta=0$ and all values of x. At large velocities of the contour, the ellipticity regions, whose dimensions can be used to approximate the initial length and position of the air–water mixing zone, appear. We denote the limiting value of the velocity V at which Eq. (18) becomes an equality by V_*. For the left x_l and right x_r boundaries of the ellipticity region, we then have the equation ($x>0$)

$$8(V_*/V)^2(0.5x^2+1+\mu a)^3=27(1+\mu a)^2x^2. \tag{19}$$

It is convenient to introduce the following notation: $Y=x^2+\lambda$, $\lambda=2(1+\mu a)$, and $æ=27\lambda^2V^2/4V_*^2$ by

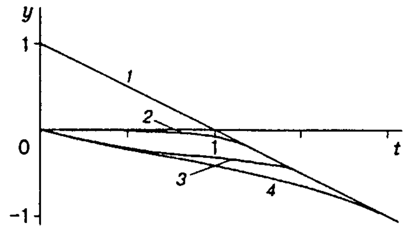

Fig. 2

means of which we can rewrite Eq. (19) as follows: $Y^3 - æY + æ\lambda = 0$. This cubic equation yields

$$\frac{x_l}{\sqrt{\lambda}} = \sqrt{3\frac{V}{V_*}\sin\left(\frac{\pi}{6} - \frac{1}{3}\arccos\frac{V_*}{V}\right) - 1}, \qquad \frac{x_r}{\sqrt{\lambda}} = \sqrt{3\frac{V}{V_*}\sin\left(\frac{\pi}{3} - \frac{1}{3}\arccos\frac{V_*}{V}\right) - 1}. \tag{20}$$

In particular, for $0 < (V_*/V - 1) \ll 1$, we obtain

$$\frac{x_l}{\sqrt{\lambda}} = \frac{1}{\sqrt{2}} - \frac{1}{2}\sqrt{3\Big(\frac{V}{V_*} - 1\Big)} + O\Big(\Big|\frac{V_*}{V} - 1\Big|\Big), \qquad \frac{x_r}{\sqrt{\lambda}} = \frac{1}{\sqrt{2}} + \frac{1}{2}\sqrt{3\Big(\frac{V}{V_*} - 1\Big)} + O\Big(\Big|\frac{V_*}{V} - 1\Big|\Big).$$

Thus, for $V < V_*$ the initial data (11) and (12) lie in the hyperbolicity region. With $V = V_*$, this property is violated at the point with coordinate $x_* = \sqrt{\lambda/2} = \sqrt{1 + \mu a}$. When $V \geqslant V_*$, an explosive growth of the ellipticity region occurs. Note that, for small $(V - V_*)/V_*$, it expands symmetrically with respect to the point x_*. For $V/V_* \to \infty$, we have

$$\frac{x_l}{\sqrt{\lambda}} \approx \frac{2}{3^{3/2}}\Big(\frac{V_*}{V}\Big), \qquad \frac{x_r}{\sqrt{\lambda}} \approx \frac{3^{3/4}}{2^{1/2}}\Big(\frac{V}{V_*}\Big)^{1/2}.$$

For sufficiently large velocities of body motion (approximately several meters per second), the ellipticity region occupies almost the entire flow region. This indicates that after the impact stage during which the incompressible-fluid model is not applicable, a mixing layer, whose parameters should enter the initial conditions for Eq. (8), is formed. Proceeding from the assumption that the thickness of the mixing layer increases with an increase in the velocity $U^+(x,0) - U^-(x,0)$, one can state that this thickness is equal to zero on the symmetry axis where the equation degenerates, grows as x increases, reaches its maximum value, and decays slowly for $x \to \infty$.

No-Gravity Approximation. It is important to note that within the no-gravity approximation $\alpha = 0$, when the initial conditions lie in the ellipticity region, deformations of the interface near the symmetry axis ($x = 0$) can be determined in a rather simple way and independently of the flow in the entire region. We consider (8) for $x = 0$. To do this, we divide both parts of the equation by h_t and pass to the limit for $x \to +0$. We obtain the ordinary differential equation relative to a new desired function $z(t) = \zeta(0,t)$

$$g_1(z,t)z'' - 2g_2(z,t)(z')^2 + g_3(z)z' = g_4(z). \tag{21}$$

Here $g_1(z,t) = (1 - z - t)(z + a)(1 - t + a\mu - z(1 - \mu))$, $g_2(z,t) = (1 - z - t)^2 - \mu(z + a)^2$, $g_3(z) = 4\mu(z + a)^2$, and $g_4(z) = -2\mu(z + a)^2$. The initial conditions for (21) follow from (11) and are of the form

$$z(0) = 0, \quad z'(0) = -a\mu/(1 + a\mu).$$

The variation in the gap between the interface and the top of a body approaching this interface $1 - t - z(t)$ for $\alpha = 0$ and $\mu = 0.8$ is shown by curve 4 of Fig. 2. This curve depends on the fluid densities and is not dependent on the body velocity. For small values of μ, the corresponding curve almost coincides with the Ox axis until $t = 1$.

Model with Allowance for Momentum Exchange between the Layers. If the type of Eq. (8) changes with time, the problem becomes nonevolutionary on some sections of the x axis, and it is impossible to continue a numerical simulation of the flow on the basis of this equation. The flow changes in such a way that Eq. (8) becomes invalid from the physical viewpoint. A model that makes it possible to describe such flows was proposed by Liapidevskii [4]. To apply this model, we define the Richardson number

$$\mathrm{Ri}(x,t) = \frac{\alpha(1-\mu)}{\mu}\,\frac{H_+ + \mu H_-}{(U^+ - U^-)^2}$$

and supplement the formulation of problem (8)–(12) by the restriction

$$\mathrm{Ri}\,(x,t) \geqslant N, \tag{22}$$

where N is the parameter of the process ($N > 1$). Formula (16) gives $\Delta(x,t) = \mu(H_+H_-)^5(U^+ - U^-)^2(\mathrm{Ri}-1)$. Hence, with allowance for (22), we have $\Delta > 0$ for all values of x and t up to the moment of contact of the solid surface with the layer interface when $H_+ = 0$. If the initial data (11) lie within the hyperbolicity region [$\Delta(x,0) > 0$], the solution of problem (8)–(12) can be constructed numerically up to t_1 at which $\mathrm{Ri}\,(x_1,t_1) = N$ at a certain point x_1. Moreover $\mathrm{Ri}\,(x,t) > N$ for $t < t_1$ and $\mathrm{Ri}(x,t_1) > N$ for $x \neq x_1$. For $t > t_1$, there is an interval $x \in [x_l(t), x_r(t)]$ inside which $\mathrm{Ri}\,(x,t) \equiv N$; note that $x_l(t_1) = x_r(t_1) = x_1$. Outside this interval, $\mathrm{Ri} > N$, and the calculation can be performed using the initial model (8)–(12). This approach is valid only if $dx_l/dt < 0$ and $dx_r/dt > 0$, which is fulfilled as early as at the initial stage of formation of the indicated interval.

Inside the specific zone, it is assumed that $\mathrm{Ri}\,(x,t) = N$, which leads to the following equation with first-order partial derivatives:

$$N\Big(\frac{x - h_t}{f(x) - t + 1 + h_x} - \frac{h_t}{a - h_x}\Big)^2 = \frac{\alpha(1-\mu)}{\mu}\,[f(x) - t + 1 + h_x(1-\mu) + \mu a]. \tag{23}$$

The initial conditions for Eq. (23) follow from the requirement for matching solutions in each zone. Once the problem has been solved, we find the interface shape in the specific zone and the velocities in each layer according to formulas (6). However, it is evident that the momentum equation (3) is not satisfied in each layer because of the momentum transfer between the layers [4]. Therefore, one should use more general relations to find the pressure: we assume that the integral law of conservation of momentum holds true in each volume $\Omega(\tilde{x},t) = \{x, y : \tilde{x} < x < x_r(t),\ -a < y < f(x) + 1 - t\}$, where $x_l(t) \leqslant \tilde{x} \leqslant x_r(t)$. The pressure varies with depth in accordance with the hydrostatic law and is continuous at the interface, from which follows that $p^+(x,y,t) = -\alpha y + \Gamma(x,t)$ and $p^-(x,y,t) = -\alpha y + \mu\Gamma(x,t) + \alpha(1-\mu)\zeta(x,t)$. The integral law of conservation of momentum for the volume $\Omega(\tilde{x},t)$ makes it possible to obtain the formula for the function $\Gamma(x,t)$

$$\Gamma(\tilde{x},t) = \Gamma(x_r,t) + \frac{1}{\mu}\,(H_+(\tilde{x},t) - H_-(\tilde{x},t))^{-1}\Big\{(1-\mu)\int\limits_{\tilde{x}}^{x_r(t)} h_{tt}(x,t)\,dx + \frac{Q_-^2(x_r,t)}{H_-(x_r,t)}$$
$$- \frac{Q_-^2(\tilde{x},t)}{H_-(\tilde{x},t)} + \mu\Big[\frac{Q_+^2(x_r,t)}{H_+(x_r,t)} - \frac{Q_+^2(\tilde{x},t)}{H_+(\tilde{x},t)}\Big] + \mu\,\frac{\alpha}{2}\,(y_b^2(\tilde{x},t) - y_b^2(x_r,t))$$
$$- \frac{\alpha}{2}\,(1-\mu)(\zeta^2(\tilde{x},t) - \zeta^2(x_r,t)) + \alpha a(1-\mu)(\zeta(x_r,t) - \zeta(\tilde{x},t))\Big\},$$

where $\Gamma(x_r(t),t)$ can be found from the momentum equation in the half-band $x > x_r(t)$.

Appearance of a Specific Zone. With separation of the zones of momentum exchange between the layers, problem (8)–(12) and (23) formulated above is rather complicated, because the positions of the boundaries of specific zones are not known beforehand and should be determined together with the solution of the problem from some additional conditions. To formulate these conditions, we consider a particular case of the motion of a wedge [$f(x) = |x|$ and $L = h_+/k$, where $k \ll 1$] toward the interface of two media. The wedge velocity is constant and is such that $\mathrm{Ri} = 1$ only at two points $x = \pm x_*$ of the Ox axis at the initial moment $t = 0$. Calculations, which are similar to those performed in derivation of Eq. (19), ensure $x_* = \lambda$, where $\lambda = 2(1 + \mu a)$. The wedge velocity V_* must be such that $\alpha = 8\mu(1-\mu)/(27\lambda)$. The initial conditions (11) and (12) and the determination of the Richardson number show that $\mathrm{Ri}\,(x,0) = (2x + \lambda)^3 x^{-2}(27\lambda)^{-1}$, $\mathrm{Ri}\,(x_*,0) = 1$, $(\partial\mathrm{Ri}/\partial x)(x_*,0) = 0$, and $(\partial^2\mathrm{Ri}/\partial x^2)(x_*,0) = 2/(3\lambda^2)$. Figure 3 shows the graph of the function

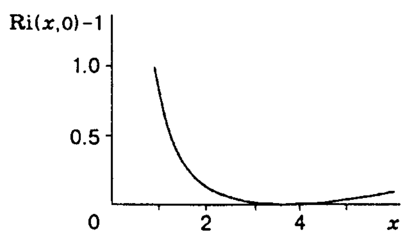

Fig. 3

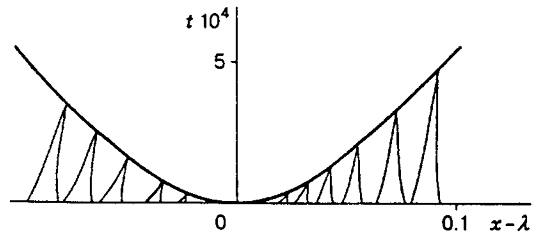

Fig. 4

$\mathrm{Ri}\,(x,0) - 1$ for $\mu = 0.8$ and $a = 1$. Equation (8) gives

$$h_{tt}(x_*,0) = \frac{8\mu}{81\lambda^2}\,[a(5\lambda+2) - (\lambda-2)(\lambda+1)],$$

from which follows

$$\frac{\partial \mathrm{Ri}}{\partial t}(x_*,0) = -\frac{R}{3\lambda^2}, \quad R = \frac{1}{9}\,[10(1-\mu) + \lambda(22-\mu)].$$

The expansion of $\mathrm{Ri}\,(x,t)$ in integer powers of $x-\lambda$ and t in the vicinity of the point of first appearance of the specific zone ($x = \lambda$, $t = 0$) can be written as

$$\mathrm{Ri}\,(x,t) = 1 + \frac{1}{3\lambda^2}\,[(x-\lambda)^2 - 2\,Rt] + \dots\,.$$

Hence, the curve $\mathrm{Ri} = 1$ is parabolic in the neighborhood of the point $x = \lambda$, $t = 0$:

$$t = \frac{1}{2R}\,(x-\lambda)^2, \tag{24}$$

where R is the radius of curvature at the parabola top. It is worth noting that R tends to the finite limit for $\mu \to 0$, which is equal to 6.

Equation (8) is of hyperbolic type in a narrow gap between the parabola (24) and the Ox axis for $|x-\lambda| \ll 1$. We write the corresponding equation of characteristics in the form

$$\frac{dx}{dt} = U^- + \frac{U^+ - U^-}{H^+ + \mu H^-}\left\{\mu H_- \pm \sqrt{\mu H_+ H_-}\sqrt{\mathrm{Ri}-1}\right\}. \tag{25}$$

Two characteristics arrive at each point of the boundary of the specific zone. They have the same slope at the boundary of the hyperbolicity zone

$$\left(\frac{dx}{dt}\right)\bigg|_{\mathrm{Ri}=1} = \frac{U^-H^+ + \mu U^+H^-}{H^+ + \mu H^-} > 0, \tag{26}$$

as follows from (25). In the general case, the normal derivative of the function $\mathrm{Ri}\,(x,t)$ on the curve $\mathrm{Ri} = 1$ is different from zero. Because of this, with distance from the boundary of the hyperbolicity region, the gap between the characteristics increases proportionally to the distance to the 3/2 power. Near the point $x = \lambda$, $t = 0$ Eq. (25) can be replaced by the approximate one

$$\frac{dx}{dt} = b_1 - b_2(x-\lambda) \pm b_3\sqrt{(x-\lambda)^2 - Rt} + \dots \tag{27}$$

$$\left[b_1 = \frac{2}{9\lambda}\,[2\mu(1+\lambda) + \lambda - 2], \quad b_2 = \frac{4\mu}{27\lambda^2}\,(a+3), \quad b_3 = \frac{4(1-\mu)}{9\sqrt{3}\lambda^2}\,\sqrt{\mu a(1+\lambda)}\right].$$

Figure 4 shows the qualitative behavior of the characteristics determined by Eq. (27) for $\mu = 0.8$, $a = 1$, $|x-\lambda| < 0.1$, and $0 < t < 6 \cdot 10^{-4}$.

Taking the above analysis into account, one can describe the evolution of the process of approach of the wedge to the layer interface as follows. Specific zones appear at the points $x = \pm\lambda$ at the initial moment and expand very rapidly. For the external boundaries of these zones $[x = \pm x_r(t)]$, one can indicate a moment t_r at which the fluid flow is not dependent on the processes occurring in the specific zones in the region $|x| > x_r(t)$, $0 < t < t_r$. For $t < t_r$, the characteristics arrive at the curves $x = \pm x_r(t)$ from the Ox axis and determine the solutions on them.

The characteristics always reach the internal boundaries of specific zones $[x = x_l(t)]$. It is noteworthy that the position of the internal boundaries of specific zones for $t > 0$ is not dependent on the processes proceeding in these zones. The fluid flow (but not the pressure) in the internal domain $[|x| < x_l(t)]$ can, therefore, be found independently.

For $N = 1$, $\alpha = 8\mu(1-\mu)/(27\lambda)$, and $\lambda = 2(1+\mu a)$, the characteristics of Eq. (23) have the positive slope

$$\frac{dx}{dt} = \frac{U^+H^- + U^-H^+}{H^+ + H^-} + \frac{1}{2}(1-\mu)\frac{H^+H^-(U^+ - U^-)}{(H^+ + H^-)(H^+ + \mu H^-)}. \tag{28}$$

It is seen from Eq. (28) that at least at the initial stage of growth of specific zones, the solutions for $x > x_r(t)$ and $x_l(t) < x < x_r(t)$ are smoothly conjugate. With $H^+ \to 0$ the characteristics (25) and (28) have the same slope equal to $U^+(x,t)$.

Numerical Results. The initial boundary-value problem (8)–(12) with limitation (22) is not a classical one. In view of this, we first performed its numerical study and clarified the basic features of the process of approach of a shallow body to the interface of two media. We considered two cases: (1) a water–incompressible gas system with density ratio $\mu = 0.01$ (the gas density is 10 times greater than the air density); (2) a water-kerosene system with $\mu = 0.8$. The body velocity was equal to 5 cm/sec in the first case and 5 mm/sec in the second one. In all calculations, $h_- = 2$ cm and $h_+ = 1$ cm, and the body has a parabolic contour with a 100-m radius of curvature at its top. Here $L = 1$ m, $a = 2$, and $\varepsilon = 0.01$. Note that, for these values of the parameters, condition (14) is not satisfied, and, hence, surface-tension forces can be ignored and one can set $\beta = 0$ in Eq. (8).The numerical results are given in dimensionless variables, which were introduced above. The distance from 0 to 1 corresponds to 1 cm in the vertical direction and 1 m in the horizontal direction.

To solve problem (8)–(12) in the hyperbolicity region of the plane x, t, we employed the method of characteristics for the functions $h_x(x,t)$ and $h_t(x,t)$. On the characteristics with the slope

$$\frac{dt_{1,2}}{dx} = \frac{B \pm \sqrt{\Delta(x,t)}}{C}$$

the conditions

$$A\frac{\partial h_t}{\partial \tau} + \frac{dt_{1,2}}{dx}C\frac{\partial h_x}{\partial \tau} = D,$$

which determine $h_x(x,t)$ and $h_t(x,t)$ at the point of intersection of the characteristics, are satisfied.

At the beginning of the body motion, the Ri and, hence, $\Delta(x,t)$ values are large, and the families of characteristics have different slopes. On approaching the specific zone, $\Delta(x,t)$ decreases, and the slopes of the characteristics become closer. The characteristics are parallel to each other at the boundary of the specific zone $[\Delta(x,t) = 0]$, which does not allow us to continue calculations with the Ri values close to 1. As soon as the Ri value becomes equal to the chosen N at a certain point, one characteristic of the first-order differential equation (23) (see, e.g., [9, Sec. 10.2]) is constructed starting from this point, along which the quantities h_x and h_t are transported. It is easy to show that at high flow velocities in the upper layer, the slopes of the characteristics of Eqs. (8) and (23) coincide at the boundary of the specific zone.

Figure 2 shows the mutual location of the body top (curve 1) and the central point of the interface with $\mu = 0.01$ (curve 2) and $\mu = 0.8$ (curves 3 and 4), where curve 3 and curves 2 and 4 correspond to a weightless fluid and that in a gravity field, respectively.

Figure 5 shows the calculation results for the first case. Clearly, the presence of the upper layer does not exert a considerable effect on interface deformation. Equation (8) is of a hyperbolic before $t_1 = 0.97$. At the moment $t_1 = 0.97$, we have $\Delta(x_1, t_1) = 0$ for the first time for $x_1 = 0.35$, and the type of Eq. (8) changes at this point.

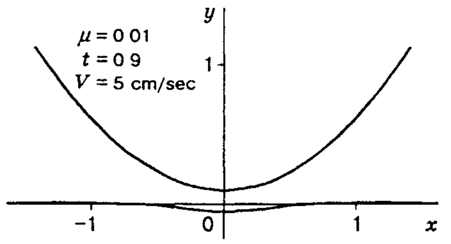

Fig. 5

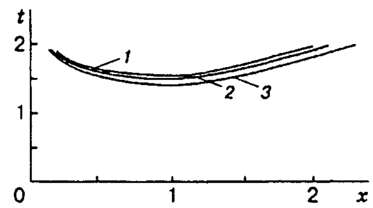

Fig. 7

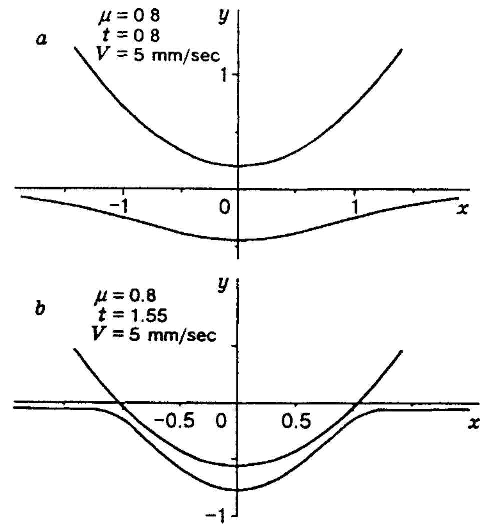

Fig. 6

In the second case, Eq. (8) is of a hyperbolic type for $t < 1.58$. The shape of the boundary and the position of the contour at $t = 0.8$ and 1.55 are shown in Fig. 6a and b, respectively. Figure 6b demonstrates that within the interval $0.75 < x < 0.95$ at $t = 1.55$, the upper layer becomes narrower, which leads to an increase in the flow velocity in this section of this layer and, as a consequence, to a change of the type of Eq. (8). The hyperbolicity of the equation is violated at $x = 0.93$ for the first time.

In the second case, Fig. 7 in the plane (x, t) shows curves 1–3 on which $\mathrm{Ri} = 5$, 10, and 15. For $x = 0$ and $x \to \infty$, the difference in the velocities in the layers equals zero and, hence, $\mathrm{Ri} = \infty$. Near the point $x = 0$, for $t > 1.8$ the variation in Ri is significant as x or t changes little. Equation (8) is of hyperbolic type below the curve $\mathrm{Ri} = 1$ and of elliptic type above this curve. The curves $\mathrm{Ri}(x, t) = N$ ($N \leqslant 1$) are spacelike. Two characteristics of Eq. (8) that determine the solution completely arrive at each point of these curves from the hyperbolicity region. The characteristic of Eq. (23) which remains within the zone on the time interval considered comes out of each point of the curves $\mathrm{Ri} = N$ to the specific zone.

Figure 8 shows the numerical results for the water–kerosene system according to the model of [4], for which Eq. (23) with $\mathrm{Ri} = 5$, 10, and 15 is satisfied (curves 1–3; curve 4 corresponds to the position of the body). It is seen that as the limiting Ri value decreases, the upper-fluid layer in the specific zone becomes thinner, and the elevation of the interface grows and then abruptly decreases. The limiting shape of the interface as $\mathrm{Ri} \to 1$ is likely to have an almost vertical slope in the vicinity of $x = x_r(t)$.

The capillary forces ($\beta \neq 0$) give rise to smoothing the interface shape. With small β, these forces can be ignored in a first approximation everywhere, except for zones in which the interface curvature is substantial. Such zones should be distinguished in the course of a numerical analysis of the problem and the "inner" asymptotics of the solution inside these zone have to be constructed. Figure 8 shows, in particular, that the interface slopes do not match each other at the external boundary of the specific zones. Therefore, near the boundaries of the momentum-transfer zones, capillary forces should be taken into account, independently of the β value. The small value of β determines the dimensions of the neighborhood of the boundary of a specific zone in which surface-tension forces are of importance. If β is of the order of unity, capillary forces are determining in the entire flow region. In the latter case, Eq. (8) is of parabolic type and is similar to the equation of beam deflection in the presence of tensile efforts at its ends. This case is of great interest and calls for separate consideration.

Flow in a Narrow Layer beneath a Body. The process considered ends at the moment of contact of the body surface with the layer interface if such a contact occurs, or at the moment of contact of the body with the bottom if the upper layer has a nonzero thickness for $t < 1 + a$. In the calculations performed for a parabolic contour, the upper layer exists for all times. This layer becomes thinner with time, following the

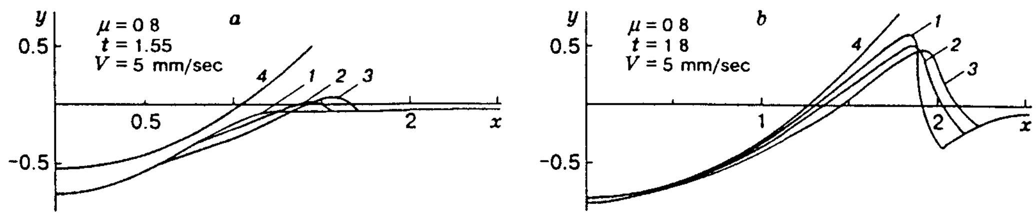

Fig. 8

shape of the body. This allows us to use an approximate scheme of description of the process: within the section $|x| < \hat{x}$, where $H_+ \ll H_-$, we ignore the presence of the upper layer to evaluate the flow parameters in the lower layer in a first approximation. Thus, we have

$$U_-(x,t) \approx \frac{x}{a - t + 1 + f(x)}, \quad H_-(x,t) \approx a + 1 + f(x) - t, \quad h_{xx}(x,t) \approx -f'(x).$$

The dynamic condition at the layer interface (7) leads to the following equation relative to $U_+(x,t)$ in the hyperbolicity region in $|x| < x_l(t)$:

$$U_t^+ + U^+ U_x^+ = \frac{1}{\mu} F(x,t) \tag{29}$$

$$\Big[F(x,t) = \frac{2}{x} U_-^2(x,t) - \frac{1}{x} f'(x) U_-^3(x,t) + \alpha(1-\mu) f'(x)\Big].$$

For a parabolic contour $[f(x) = x^2/2]$, it can be directly verified that $F(x,t) \geqslant 0$. The liquid particles of the upper layer move along the trajectories $dx/dt = U_+(x,t)$, and their velocities change according to the law

$$\frac{dU_+}{dt} = \frac{F(x,t)}{\mu},$$

where d/dt is the differentiation operator along the trajectory. Hence, the liquid particles accelerate with distance from the symmetry axis.

In the momentum-transfer zone between the layers $[x_l(t) < |x| < x_r(t)]$, condition (7) is replaced by the equality $\mathrm{Ri}(x,t) = N$ which yields

$$U^+(x,t) = U^-(x,t) + \sqrt{\alpha(1-\mu)H_-/N}, \tag{30}$$

where the plus sign is chosen based on physical reasoning: the velocity in the upper layer is larger than in the lower one. The position of the internal boundary of a specific zone $[x = x_l(t)]$ is found from the condition that the velocities that are determined by formulas (29) and (30) are equal at this boundary. The numerical flow simulation wihin the original model (see Fig. 8) shows that $\hat{x}(t) < x_r(t)$. At this stage, the specific zone's external boundary $[x = x_r(t)]$ is calculated according to model (8)–(12).

In the model of [4] with momentum exchange between the layers, mass transfer and mixing were not taken into account, and, therefore, the approximate equation for the thickness of the upper layer H^+ holds:

$$H_t^+ + (U^+ H^+)_x = 0. \tag{31}$$

In particular, relations (29) and (31) give $H^+(0,t) \approx C_1(a + 1 - t)^q + \ldots$ as $t \to a + 1$ with constant C_1, which can be determined from the matching condition of the approximate and numerical solutions, and $q = \sqrt{1/4 + 2/\mu} - 1/2$.

Thus, the model considered can be recommended for analysis of the motion of shallow bodies near the interface boundaries. The upper layer exists for all times within the model, but one can distinguish a region in the vicinity of the body top within which this layer is very thin. The dimensions of this region depend little on the Richardson limiting value (Fig. 8). The presence of a thin upper layer provides a negligible contribution to the fluid flow in the lower layer and to the pressure distribution along the body surface [10].

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